A trick around Fibonacci, Lucas and Chebyshev

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Abstract

In this article, we present a trick around Fibonacci numbers which can be found in several magic books. It consists in computing quickly the sum of the successive terms of a Fibonacci-like sequence. We give explanations and extensions of this trick to more general sequences. This study leads us to interesting connections between Fibonacci, Lucas sequences and Chebyshev polynomials.

1 Introduction

We describe a famous magic trick which can be found in many magic treatises; see, e.g., [3], [4], [5], [6]. A magician asks you to choose two integers (not too complicated since you will have to compute several sums) in your head. Add them and write the result for instance on a paper that the magician does not see for the time being. Next add the number you just wrote to the second number you have chosen and write the result on your paper. Next add the two numbers you wrote and write the result, and so on. Each time, you add the two last numbers you obtained and you write the result. Repeat this procedure until you have written 8 numbers on your paper. Finally add all these numbers including the two first numbers you chose: this is the sum of 10 numbers. Now ask the magician to look at your paper. Without knowing the two first integers you chose, he will quickly obtain the sum by using only one term written on the paper. Indeed, we claim that this is 11 times the fifth term you wrote (or the fourth term starting from the last one you wrote). You have certainly understood that you were dealing with the famous Fibonacci numbers (with general initial conditions)...

The explanation is the following one: you choose two integers x_0 and x_1 (actually the trick holds for any complex numbers) and you successively write $x_2 = x_0 + x_1$, $x_3 = x_1 + x_2$, $x_4 = x_2 + x_3$, $x_5 = x_3 + x_4$, $x_6 = x_4 + x_5$, $x_7 = x_5 + x_6$, $x_8 = x_6 + x_7$ and $x_9 = x_7 + x_8$. All these numbers can be expressed by means of x_0 and x_1:

$$x_2 = x_0 + x_1,\ x_3 = x_0 + 2x_1,\ x_4 = 2x_0 + 3x_1,\ x_5 = 3x_0 + 5x_1,$$
$$x_6 = 5x_0 + 8x_1,\ x_7 = 8x_0 + 13x_1,\ x_8 = 13x_0 + 21x_1,\ x_9 = 21x_0 + 34x_1.$$

Finally, you compute the sum of all the numbers from x_0 to x_9:

$$S = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9.$$

You can easily check that $S = 55x_0 + 88x_1 = 11x_6$. A simpler trick consists in writing only 4 numbers (x_2, x_3, x_4, x_5) and computing the sum of 6 numbers:

$$S' = x_0 + x_1 + x_2 + x_3 + x_4 + x_5.$$

This time, we have $S' = 8x_0 + 12x_1 = 4x_4$. We claim that the sum S' is 4 times the third term you wrote.

The relationships $S = 11x_6$ and $S' = 4x_4$, which hold for any choice of x_0 and x_1, are the starting point of our investigation. In this article, we address two problems:

- Is it possible to find in a Fibonacci sequence with general initial conditions other couple of integers (n, m) such that the sum of the n first terms is proportional to the mth term, possibly through a simple ratio (e.g., an integer)?

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- Is it possible to observe a similar property for more general sequences satisfying a second order linear recurrence relation?

Fibonacci numbers consist without any doubt of one of the most well-known and important sequence of numbers in Mathematics, Life and Nature! It enjoys so many remarkable, fascinating, funny properties that it is the source of a lot of entertainment. Our problem relies on certain worthwhile properties concerning the sum of Fibonacci numbers.

Although starting from a classical trick used in magic, our aim is not to invent more spectacular magic tricks but to introduce well-known or less-known interesting relationships between different and important mathematical objects.

Throughout the paper, $\mathbb{N}$ will denote the set of non-negative integers, that is $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{R}$ the set of real numbers and $\mathbb{C}$ that of complex numbers.

The paper is organized as follows. In Section 2, we recall and exhibit several features of classical Fibonacci and Lucas numbers (satisfying the recurrence $x_{n+2} = x_{n+1} + x_n$) and we give a positive answer to the first question we addressed above. In Section 3, we consider more general Fibonacci-like sequences, those satisfying the second order linear recurrence $x_{n+2} = ax_{n+1} + bx_n$. For those corresponding to the value $b = -1$, we are able to answer positively to the second question. Our analysis involves interesting connections between such sequences and the famous Chebyshev polynomials which are described in Section 4. Concerning the value $b = 1$ (and $a \neq 1$), we are able to answer partially to the second question. More precisely, we get a proportionality identity only for certain numbers a. This case is related to the family of the Fibonacci polynomials which are also described in Section 4. In Section 5, we sum up the results obtained in this article. Finally, we include in an appendix some additional and technical informations concerning the case where $b = 1$.

In order to facilitate the reading of the article and to make it self-contained and broadly accessible, we report many details. Nonetheless, we assume that the reader is familiar with linear recursive sequences. Since there is a huge literature on Fibonacci, we have chosen to give a concise and uniform list of references based on electronic resources like Wikipedia and Wolfram.

2 The classical Fibonacci and Lucas sequences

2.1 Recall of the definitions and certain properties

In this section, we recall definitions and some well-known facts on the classical Fibonacci and Lucas numbers; we refer the reader to [1], [9], [13] and [15]. They are defined by

$$\mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1\ \text{ and }\ \mathcal{F}_{n+2} = \mathcal{F}_{n+1} + \mathcal{F}_n\ \text{ for } n \in \mathbb{N},$$
$$\mathcal{L}_0 = 2,\ \mathcal{L}_1 = 1\ \text{ and }\ \mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n\ \text{ for } n \in \mathbb{N}.$$

We write the first terms:

$$\mathcal{F}_0 = 0,\ \mathcal{F}_1 = 1,\ \mathcal{F}_2 = 1,\ \mathcal{F}_3 = 2,\ \mathcal{F}_4 = 3, \mathcal{F}_5 = 5,\ \mathcal{F}_6 = 8,\ \mathcal{F}_7 = 13,\ \mathcal{F}_8 = 21,\ \mathcal{F}_9 = 34 \dots$$
$$\mathcal{L}_0 = 2,\ \mathcal{L}_1 = 1,\ \mathcal{L}_2 = 3,\ \mathcal{L}_3 = 4,\ \mathcal{L}_4 = 7, \mathcal{L}_5 = 11,\ \mathcal{L}_6 = 18,\ \mathcal{L}_7 = 29,\ \mathcal{L}_8 = 47,\ \mathcal{L}_9 = 76 \dots$$

Let us introduce the famous golden number $\varphi_1 = \left(1 + \sqrt{5}\right)/2$ together with $\varphi_2 = \left(1 - \sqrt{5}\right)/2$. The numbers φ_1 and φ_2 are the solutions of the quadratic equation $\varphi^2 = \varphi + 1$. They satisfy $\varphi_1 + \varphi_2 = 1$ and $\varphi_1 \varphi_2 = -1$. Binet's formula supplies explicit expressions for $\mathcal{F}_n$ and $\mathcal{L}_n$:

$$\mathcal{F}_n = \frac{\varphi_1^n - \varphi_2^n}{\varphi_1 - \varphi_2}\ \text{ and }\ \mathcal{L}_n = \varphi_1^n + \varphi_2^n.$$

We first state a property concerning the sum of the Fibonacci numbers. Set $\mathcal{S}_0 = 0$ and $\mathcal{S}_n = \sum_{k=0}^{n-1} \mathcal{F}_k$ for any integer $n \geq 1$. The sum $\mathcal{S}_n$ can be immediately evaluated by performing a telescopic sum: indeed, we have $\mathcal{S}_n = \sum_{k=0}^{n-1} (\mathcal{F}_{k+2} - \mathcal{F}_{k+1})$, which can be simplified into

$$\mathcal{S}_n = \mathcal{F}_{n+1} - 1.$$

In relation to the trick described in the introduction, we mention an interesting relationship between Fibonacci and Lucas numbers: for any $p, q \in \mathbb{N}$ such that $p \geq q$,

$$\mathcal{F}_{p+q} + (-1)^q \mathcal{F}_{p-q} = \mathcal{F}_p \mathcal{L}_q. \tag{1}$$

The proof of this identity is quite elementary:

$$\begin{aligned}
(\varphi_1 - \varphi_2)\mathcal{F}_p \mathcal{L}_q &= (\varphi_1^p - \varphi_2^p)(\varphi_1^q + \varphi_2^q) = (\varphi_1^{p+q} - \varphi_2^{p+q}) + (\varphi_1^p \varphi_2^q - \varphi_1^q \varphi_2^p) \\
&= (\varphi_1^{p+q} - \varphi_2^{p+q}) + (\varphi_1 \varphi_2)^q (\varphi_1^{p-q} - \varphi_2^{p-q}) = (\varphi_1^{p+q} - \varphi_2^{p+q}) + (-1)^q(\varphi_1^{p-q} - \varphi_2^{p-q}) \\
&= (\varphi_1 - \varphi_2)(\mathcal{F}_{p+q} + (-1)^q \mathcal{F}_{p-q}).
\end{aligned}$$

Let $m \in \mathbb{N}$. In particular, (1) yields for $p = q = m$ and for $p = m+1$, $q = m$, m being odd in the second case:

$$\mathcal{F}_{2m} = \mathcal{F}_m \mathcal{L}_m \quad \text{and} \quad \mathcal{F}_{2m+1} = \mathcal{F}_{m+1} \mathcal{L}_m + 1, \tag{2}$$

from which we derive

$$\mathcal{S}_{2m-1} = \mathcal{F}_m \mathcal{L}_m - 1 \quad \text{and} \quad \mathcal{S}_{2m} = \mathcal{F}_{m+1} \mathcal{L}_m.$$

We then state the following important identity for our problem.

Theorem 1 *If n is a multiple of 4 plus 2, we have the following equality:*

$$\mathcal{S}_n = \mathcal{L}_{n/2} \mathcal{F}_{n/2+1}. \tag{3}$$

We retrieve our introductory examples: $\mathcal{S}_6 = \mathcal{L}_3 \mathcal{F}_4 = 4\mathcal{F}_4$ and $\mathcal{S}_{10} = \mathcal{L}_5 \mathcal{F}_6 = 11\mathcal{F}_6$.

The identity (3) can be extended to a sum starting from $\mathcal{F}_r$ for a certain $r \in \mathbb{N}$: with the same assumption on n, (1) yields for $p = r + n/2 + 1$ and $q = n/2$:

$$\sum_{k=0}^{n-1} \mathcal{F}_{k+r} = \mathcal{F}_{n+r+1} - \mathcal{F}_{r+1} = \mathcal{L}_{n/2} \mathcal{F}_{r+n/2+1}.$$

For instance, we can find in [13] the proportionality identities:

$$\mathcal{F}_r + \mathcal{F}_{r+1} + \cdots + \mathcal{F}_{r+5} = 4\mathcal{F}_{r+4},$$
$$\mathcal{F}_r + \mathcal{F}_{r+1} + \cdots + \mathcal{F}_{r+9} = 11\mathcal{F}_{r+6}.$$

2.2 Fibonacci sequence with general initial conditions

Now, we consider a Fibonacci sequence $(x_n)_{n\in\mathbb{N}}$ with general initial conditions. We are given two complex numbers x_0, x_1 and we assume that $x_{n+2} = x_{n+1} + x_n$ for any $n \in \mathbb{N}$.

Put $S_0 = 0$ and $S_n = \sum_{k=0}^{n-1} x_k$ for any integer $n \geq 1$. This sum can be easily computed as before:

$$S_n = x_{n+1} - x_1.$$

We propose two approaches to tackle the problem of finding a proportionality identity of the form $S_n = Ax_m$ by appealing to simple arguments of linear algebra. We know that the set of Fibonacci sequences (with general initial conditions) is a vectorial space $\mathcal{E}$ of dimension 2. We use decompositions of the sequence $(x_n)_{n\in\mathbb{N}}$ over two appropriate bases of $\mathcal{E}$. Our aim is not to find all m but only to derive a convenient integer m between 1 and n.

First approach

We consistently put $\mathcal{F}_{-1} = 1$, so that the sequence $(\mathcal{F}_{n-1})_{n\in\mathbb{N}}$ is a Fibonacci sequence with initial conditions $\mathcal{F}_{-1} = 1$ and $\mathcal{F}_0 = 0$. It is clear that the family $\{(\mathcal{F}_{n-1})_{n\in\mathbb{N}}, (\mathcal{F}_n)_{n\in\mathbb{N}}\}$ is a basis of $\mathcal{E}$ and that $(x_n)_{n\in\mathbb{N}}$ can be decomposed as

$$x_n = x_0 \mathcal{F}_{n-1} + x_1 \mathcal{F}_n, \quad n \in \mathbb{N}.$$

Therefore, concerning the sum S_n,

$$S_n = x_{n+1} - x_1 = x_0 \mathcal{F}_n + x_1(\mathcal{F}_{n+1} - 1).$$

In particular, referring to (2), for any integer $m \geq 1$,

$$S_{2m-2} = x_0 \mathcal{F}_{2m-2} + x_1(\mathcal{F}_{2m-1} - 1) = \mathcal{L}_{m-1}(x_0 \mathcal{F}_{m-1} + x_1 \mathcal{F}_m) = \mathcal{L}_{m-1} x_m.$$

This approach directly supplies an answer to our problem: for any positive integer n which is a multiple of 4 plus 2, we have found an integer $m = n/2 + 1$ and a proportionality factor $\mathcal{A} = \mathcal{L}_{n/2}$ such that $S_n = \mathcal{A} x_m$.

Theorem 2 *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of complex numbers satisfying the recurrence relation $x_{n+2} = x_{n+1} + x_n$ for any $n \in \mathbb{N}$. For any positive integer n which is a multiple of 4 plus 2, the following identity holds:*

$$S_n = \mathcal{L}_{n/2} x_{n/2+1}. \tag{4}$$

Identity (4) is quite analogous to (3). Moreover, it is worth noting that the factor $\mathcal{A} = \mathcal{L}_{n/2}$ is a positive integer which does not depend on the initial conditions, according to the goal of our investigation. This approach essentially hinges on the particular form of the recurrence relation leading to a telescopic sum, which is quite restrictive. In order to extend Theorem 2 to more general sequences, we propose a more constructive approach which will works in a wider context.

Second approach

We use another basis of $\mathcal{E}$: $\{(\varphi_1^n)_{n \in \mathbb{N}}, (\varphi_2^n)_{n \in \mathbb{N}}\}$. Then $(x_n)_{n \in \mathbb{N}}$ can be decomposed as

$$x_n = \alpha_1 \varphi_1^n + \alpha_2 \varphi_2^n, \quad n \in \mathbb{N},$$

where α_1 and α_2 are two parameters characterized by the equalities $\alpha_1 + \alpha_2 = x_0$ and $\alpha_1 \varphi_1 + \alpha_2 \varphi_2 = x_1$. Actually, it is not necessary to determine explicitly α_1 and α_2. With these settings at hand, the sum S_n can be evaluated as follows:

$$S_n = \alpha_1 \sum_{k=0}^{n-1} \varphi_1^k + \alpha_2 \sum_{k=0}^{n-1} \varphi_2^k = \alpha_1 \frac{\varphi_1^n - 1}{\varphi_1 - 1} + \alpha_2 \frac{\varphi_2^n - 1}{\varphi_2 - 1}.$$

We would like to relate S_n to one term x_m of the sequence through a simple proportionality identity independently on α_1 and α_2. More precisely, is it possible to find, for any integer n, a factor $\mathcal{A}$ and an integer m such that $S_n = \mathcal{A} x_m$ for any choice of x_0, x_1?

This is equivalent to finding, for any n, numbers $\mathcal{A}$ and m such that

$$\forall \alpha_1, \alpha_2 \in \mathbb{C}, \ \alpha_1 \frac{\varphi_1^n - 1}{\varphi_1 - 1} + \alpha_2 \frac{\varphi_2^n - 1}{\varphi_2 - 1} = \alpha_1 \mathcal{A} \varphi_2^m + \alpha_1 \mathcal{A} \varphi_2^m$$

or

$$\frac{\varphi_1^n - 1}{\varphi_1 - 1} = \mathcal{A} \varphi_1^m \ \text{ and } \ \frac{\varphi_2^n - 1}{\varphi_2 - 1} = \mathcal{A} \varphi_2^m.$$

These last equalities can be rewritten as

$$\varphi_1^n - 1 = \mathcal{A}(\varphi_1^{m+1} - \varphi_1^m) \ \text{ and } \ \varphi_2^n - 1 = \mathcal{A}(\varphi_2^{m+1} - \varphi_2^m)$$

or, since $\varphi_1^2 = \varphi_1 + 1$ and $\varphi_2^2 = \varphi_2 + 1$ which entail that $\varphi_1^{m+1} - \varphi_1^m = \varphi_1^{m-1}$ and $\varphi_2^{m+1} - \varphi_2^m = \varphi_2^{m-1}$,

$$\varphi_1^n - 1 = \mathcal{A} \varphi_1^{m-1} \ \text{ and } \ \varphi_2^n - 1 = \mathcal{A} \varphi_2^{m-1}. \tag{5}$$

Hence, we are searching for $\mathcal{A}$ and m such that

$$\mathcal{A} = \frac{\varphi_1^n - 1}{\varphi_1^{m-1}} = \frac{\varphi_2^n - 1}{\varphi_2^{m-1}}. \tag{6}$$

Consequently, m solves the equation

$$\left(\frac{\varphi_1}{\varphi_2}\right)^{m-1} = \frac{\varphi_1^n - 1}{\varphi_2^n - 1}.$$

Substituting $\varphi_2 = -1/\varphi_1$ in the foregoing equation yields

$$\varphi_1^{2m-n-2} = (-1)^m \frac{\varphi_1^n - 1}{\varphi_1^n - (-1)^n}.$$

When n is odd, it is difficult to extract m and it seems hopeless to obtain an integer. When n is even instead, this is equivalent to

$$\varphi_1^{2m-n-2} = (-1)^m.$$

Since $\varphi_1 > 0$, the integer m must be even too and $n = 2m - 2$. As a result, n must be a multiple of 4 plus 2 and $m = n/2 + 1$. Next, by (6), the coefficient $\mathcal{A}$ writes

$$\mathcal{A} = \frac{\varphi_1^n - 1}{\varphi_1^{n/2}} = \varphi_1^{n/2} - \varphi_1^{-n/2} = \varphi_1^{n/2} + \varphi_2^{n/2} = \mathcal{L}_{n/2}.$$

We just retrieved the solution (4) obtained by the first approach.

Remark 1 *Let us remind that if an equality between two ratios $f := \dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ holds, then $f = \dfrac{a_1 - a_2}{b_1 - b_2}$; this is quite an elementary fact! This yields here*

$$\mathcal{A} = \frac{\varphi_1^n - \varphi_2^n}{\varphi_1^{m-1} - \varphi_2^{m-1}} = \frac{\mathcal{F}_n}{\mathcal{F}_{m-1}}$$

and we retrieve for $m = n/2 + 1$, $\mathcal{A} = \mathcal{F}_n / \mathcal{F}_{n/2} = \mathcal{L}_{n/2}$.

The second approach allows us to tackle our problem for more general sequences. We employ it in the next section.

A quite similar analysis can be carried out *mutatis mutandis* for the sequence $(x_n)_{n \in \mathbb{N}}$ defined by general initial conditions $x_0, x_1 \in \mathbb{C}$ and the recurrence relation $x_{n+2} = -x_{n+1} + x_n$ for any $n \in \mathbb{N}$. The numbers φ_1 and φ_2 have to be changed into the roots ψ_1 and ψ_2 of the equation $\psi^2 = -\psi + 1$. Explicitly, $\psi_1 = -\varphi_1$ and $\psi_2 = -\varphi_2$. The sequence corresponding to the initial values 0 and 1 consists of the numbers $(\psi_1^n - \psi_2^n)/(\psi_1 - \psi_2)$ which are nothing but $(-1)^{n-1}\mathcal{F}_n$. Since $\psi_1^{m+1} - \psi_1^m = -\psi_1^{m+2}$ and $\psi_2^{m+1} - \psi_2^m = -\psi_2^{m+2}$, the analogous of (5) is

$$\psi_1^n - 1 = -\mathcal{A}'\psi_1^{m+2} \quad \text{and} \quad \psi_2^n - 1 = -\mathcal{A}'\psi_2^{m+2}$$

from which we extract

$$\left(\frac{\psi_1}{\psi_2}\right)^{m+2} = \frac{\psi_1^n - 1}{\psi_2^n - 1}$$

or, equivalently when n is even, $\varphi_1^{2m-n+4} = (-1)^{m-1}$. We see that the integer m must be odd and $n = 2m + 4$. So, the integer n must be a multiple of 4 plus 2 and $m = n/2 - 2$. Next,

$$\mathcal{A}' = -\frac{\psi_1^n - 1}{\psi_1^{n/2}} = \psi_1^{-n/2} - \psi_1^{n/2} = \varphi_1^{n/2} + \varphi_2^{n/2} = \mathcal{L}_{n/2}.$$

We observe that $\mathcal{A}' = \mathcal{A}$. As a result, we can state the result below.

Theorem 3 *Let $(x_n)_{n \in \mathbb{N}}$ be a sequence satisfying the recurrence relation $x_{n+2} = -x_{n+1} + x_n$ for any $n \in \mathbb{N}$. For any positive integer n which is a multiple of 4 plus 2, the following identity holds:*

$$S_n = \mathcal{L}_{n/2} x_{n/2-2}.$$

As an example of a trick, for $n = 10$, you choose two numbers x_0 and x_1 and you successively write

$$x_2 = x_0 - x_1, \ x_3 = -x_0 + 2x_1, \ x_4 = 2x_0 - 3x_1, \ x_5 = -3x_0 + 5x_1,$$
$$x_6 = 5x_0 - 8x_1, \ x_7 = -8x_0 + 13x_1, \ x_8 = 13x_0 - 21x_1, \ x_9 = -21x_0 + 34x_1.$$

Finally, you compute the sum of all the numbers from x_0 to x_9: $S = x_0 + x_1 + \cdots + x_9$. You find that

$$S = -11x_0 + 22x_1 = 11x_3.$$

We guess that this trick is simple enough to be exploited in magic...

3   The second order linear recurrence

Let us fix two complex numbers a, b and let us consider the sequence $(x_n)_{n \in \mathbb{N}}$ defined by two initial conditions $x_0, x_1 \in \mathbb{C}$ and the recurrence relation $x_{n+2} = ax_{n+1} + bx_n$ for any $n \in \mathbb{N}$. When the parameters a and b are integers, $(x_n)_{n \in \mathbb{N}}$ is called a Lucas sequence; see [11] and [16].

Let us introduce r_1 and r_2 the complex roots of the quadratic equation $r^2 = ar + b$. We assume that $a^2 + 4b \neq 0$ in order to have two distinct roots ($r_1 \neq r_2$), that $b \neq 0$ and $a + b \neq 1$ in order that 0 and 1 are not roots ($r_1, r_2 \notin \{0, 1\}$). The numbers r_1 and r_2 satisfy $r_1 r_2 = -b$.

The explicit expression of x_n is then given by $x_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ where α_1 and α_2 are two parameters such that $\alpha_1 + \alpha_2 = x_0$ and $\alpha_1 r_1 + \alpha_2 r_2 = x_1$. It will not be necessary to determine α_1 and α_2. Put $S_0 = 0$ and $S_n = \sum_{k=0}^{n-1} x_k$ for $n \geq 1$. We have

$$S_n = \alpha_1 \frac{r_1^n - 1}{r_1 - 1} + \alpha_2 \frac{r_2^n - 1}{r_2 - 1}. \tag{7}$$

As previously, is it possible to find, for any integer n, a coefficient A and an integer m such that $S_n = Ax_m$ for any choice of x_0 and x_1?

3.1   Solving the problem

The problem is equivalent to finding, for any n, numbers A and m such that

$$r_1^n - 1 = A(r_1 - 1)r_1^m \quad \text{and} \quad r_2^n - 1 = A(r_2 - 1)r_2^m.$$

So, we get

$$A = \frac{r_1^n - 1}{(r_1 - 1)r_1^m} = \frac{r_2^n - 1}{(r_2 - 1)r_2^m} \tag{8}$$

which entails

$$\left(\frac{r_1}{r_2}\right)^m = \frac{r_2 - 1}{r_1 - 1} \frac{r_1^n - 1}{r_2^n - 1}.$$

Substituting $r_2 = -b/r_1$ in the foregoing equation yields

$$r_1^{2m-n+1} = (-b)^m \frac{r_1 + b}{r_1 - 1} \frac{r_1^n - 1}{r_1^n - (-b)^n}. \tag{9}$$

When $b \neq \pm 1$, Equation (9) is difficult to solve and it seems hopeless to find an integer m satisfying it; we shall not go further in this direction. We now focus precisely on the cases $b = -1$ and $b = 1$.

The case $b = -1$

Suppose that $b = -1$. The assumptions $a^2 + 4b \neq 0$ and $a + b \neq 0$ force a to be chosen different from 2 and -2. Equation (9) reads $r_1^{2m-n+1} = 1$ which implies that $n = 2m + 1$. Hence, n must be odd and $m = (n-1)/2$.

Next, we have to determine the coefficient A. Using the proportionality rule stipulating that if $\alpha := \alpha_1 = \alpha_2$, then $\alpha = [(r_1 - 1)\alpha_1 - (r_2 - 1)\alpha_2]/(r_1 - r_2)$, and the fact that $r_1 r_2 = -b = 1$, we extract from (8) the alternative form

$$A = \frac{1}{r_1 - r_2}\left(\frac{r_1^n - 1}{r_1^m} - \frac{r_2^n - 1}{r_2^m}\right) = \frac{r_1^{n-m} - r_2^{n-m}}{r_1 - r_2} + \frac{r_1^m - r_2^m}{r_1 - r_2}. \tag{10}$$

By introducing the particular sequence $(u_n)_{n \in \mathbb{N}}$ satisfying the same recurrence relation as $(x_n)_{n \in \mathbb{N}}$ with initial conditions $u_0 = 0$ and $u_1 = 1$, whose expression is $u_n = (r_1^n - r_2^n)/(r_1 - r_2)$, we see that the number A can be written as $A = u_{n-m} + u_m = u_{m+1} + u_m$, that is

$$A = u_{(n+1)/2} + u_{(n-1)/2}. \tag{11}$$

In Section 3.2, we propose a closed expression for A by means of a.

Let us have a look at the cases $a = \pm 2$ which were excluded from the previous analysis.

- When $a = 2$, x_n is of the form $x_n = \alpha_1 n + \alpha_2$ for certain parameters α_1 and α_2, $u_n = n$ for any $n \in \mathbb{N}$ and, if n is odd,

$$S_n = \alpha_1 \frac{n(n-1)}{2} + n\alpha_2 = n\left(\alpha_1 \frac{n-1}{2} + \alpha_2\right) = nx_{(n-1)/2}.$$

We find that the equality $S_n = Ax_{(n-1)/2}$ holds with $A = n$ which coincides with $u_{(n+1)/2}+u_{(n-1)/2}$.

- When $a = -2$, we similarly have $x_n = (-1)^n(\alpha_1 n + \alpha_2)$, $u_n = (-1)^{n+1}n$ for any $n \in \mathbb{N}$ and, if $n = 2m + 1$,

$$S_n = S_{2m+1} = \alpha_1 m + \alpha_2 = (-1)^m x_m = (-1)^{(n-1)/2} x_{(n-1)/2}.$$

Then $S_n = Ax_{(n-1)/2}$ with $A = (-1)^{(n-1)/2}$. This A coincides with $u_{(n+1)/2} + u_{(n-1)/2}$ and (11) holds in this case also.

We conclude as follows.

Theorem 4 *Fix a complex number a. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence satisfying the recurrence relation $x_{n+2} = ax_{n+1} - x_n$ for any $n \in \mathbb{N}$ and $(u_n)_{n\in\mathbb{N}}$ be the sequence satisfying the same relation such that $u_0 = 0$ and $u_1 = 0$. For any positive and odd integer n, the following proportionality identity holds:*

$$S_n = (u_{(n+1)/2} + u_{(n-1)/2})x_{(n-1)/2}.$$

Example 1 *Let us write the first numbers of the sequence:*

$$x_2 = -x_0 + ax_1,$$
$$x_3 = -ax_0 + (a^2 - 1)x_1,$$
$$x_4 = (-a^2 + 1)x_0 + (a^3 - 2a)x_1,$$
$$x_5 = (-a^3 + 2a)x_0 + (a^4 - 3a^2 + 1)x_1,$$
$$x_6 = (-a^4 + 3a^2 - 1)x_0 + (a^5 - 4a^3 + 3a)x_1,$$
$$x_7 = (-a^5 + 4a^3 - 3a)x_0 + (a^6 - 5a^4 + 6a^2 - 1)x_1,$$
$$x_8 = (-a^6 + 5a^4 - 6a^2 + 1)x_0 + (a^7 - 6a^5 + 10a^3 - 4a)x_1,$$
$$x_9 = (-a^7 + 6a^5 - 10a^3 + 4a)x_0 + (a^8 - 7a^6 + 15a^4 - 10a^2 + 1)x_1,$$
$$x_{10} = (-a^8 + 7a^6 - 15a^4 + 10a^2 - 1)x_0 + (a^9 - 8a^7 + 21a^5 - 20a^3 + 5a)x_1.$$

When choosing $x_0 = 0$ and $x_1 = 1$, we obtain $u_5 = a^4 - 3a^2 + 1$ as well as $u_6 = a^5 - 4a^3 + 3a$. The sum of all numbers from x_0 to x_{10} ($n = 11$) is given by

$$S_{11} = (-a^8 - a^7 + 6a^6 + 5a^5 - 11a^4 - 7a^3 + 6a^2 + 2a)x_0$$
$$+ (a^9 + a^8 - 7a^7 - 6a^6 + 16a^5 + 11a^4 - 13a^3 - 6a^2 + 3a + 1)x_1$$

The reader can easily check that

$$S_{11} = (a^5 + a^4 - 4a^3 - 3a^2 + 3a + 1)[(-a^3 + 2a)x_0 + (a^4 - 3a^2 + 1)x_1] = (u_5 + u_6)x_5.$$

The case $b = 1$

Let us now consider the case where $b = 1$. We are dealing with the recurrence relation $x_{n+2} = ax_{n+1}+x_n$. Assume that m and n are even. Equation (9) reads

$$r_1^{2m-n+1} = \frac{r_1 + 1}{r_1 - 1}. \tag{12}$$

Fix a positive integer p and choose a complex root ρ_1 of the equation $r^{p+1} - r^p - r - 1 = 0$. Set $\rho_2 = -1/\rho_1$. If we choose $a = \rho_1 + \rho_2$, the roots ρ_1 and ρ_2 satisfy $\rho_1^2 = a\rho_1 + 1$ and $\rho_2^2 = a\rho_2 + 1$. In order to use our analysis, the equation $r^2 = ar + 1$ must have two different roots, that is, we must choose ρ_1 different from $\pm i$. The numbers $\pm i$ are roots when the power p is a multiple of 4 minus 1. Since $\rho_1^p = (\rho_1 + 1)/(\rho_1 - 1)$, the number ρ_1 solves Equation (12) if and only if $\rho_1^{2m-n+1} = \rho_1^p$ which is clearly

satisfied when $n = 2m + 1 - p$, or $m = (n + p - 1)/2$. Taking into account the assumptions that m and n are even, p must be chosen *a priori* odd such that $(n + p - 1)$ be a multiple of 4.

Finally, by introducing the particular sequence $(u_n)_{n \in \mathbb{N}}$ satisfying the same recurrence relation as $(x_n)_{n \in \mathbb{N}}$ with initial conditions $u_0 = 0$ and $u_1 = 1$, it is quite easy to see that the expression $A = u_{n-m} + u_m$ for the proportionality coefficient provided in the case $b = -1$ still holds, due to the fact that m is even and $\rho_1 \rho_2 = -1$. Therefore,

$$A = u_{(n-p+1)/2} + u_{(n+p-1)/2}. \tag{13}$$

Remark 2 *The equation $r^{p+1} - r^p - r - 1 = 0$ admits at least one positive real root: indeed, it is equivalent to $r^p = \frac{r+1}{r-1}$, the roots of which are the intersection of the curves of the maps $x \mapsto x^p$ and $x \mapsto \frac{x+1}{x-1}$. More precisely, an elementary study shows that the equation has exactly one positive root (which lies in the interval $(1, 2)$) when p is even and two roots (one is positive–lying in $(1, 2.5)$–and the other is the opposite of the inverse of the first one) if p is odd.*

Remark 3 *Referring to the first equality of (10) and using the facts that m and n are even and that $\rho_1 \rho_2 = -1$, we can derive another expression of A:*

$$
\begin{aligned}
A &= \frac{1}{\rho_1 - \rho_2} \left(\frac{\rho_1^n - 1}{\rho_1^m} - \frac{\rho_2^n - 1}{\rho_2^m} \right) \\
&= \rho_1^{n/2-m} \frac{\rho_1^{n/2} - \rho_1^{-n/2}}{\rho_1 - \rho_2} - \rho_2^{n/2-m} \frac{\rho_2^{n/2} - \rho_2^{-n/2}}{\rho_1 - \rho_2} \\
&= \rho_1^{n/2-m} \frac{\rho_1^{n/2} - (-1)^{n/2}\rho_2^{n/2}}{\rho_1 - \rho_2} - \rho_2^{n/2-m} \frac{\rho_2^{n/2} - (-1)^{n/2}\rho_1^{n/2}}{\rho_1 - \rho_2} \\
&= \frac{\left(\rho_1^{n/2-m} + (-1)^{n/2}\rho_2^{n/2-m} \right)\left(\rho_1^{n/2} - (-1)^{n/2}\rho_2^{n/2} \right)}{\rho_1 - \rho_2} \\
&= \frac{\left(\rho_1^{m-n/2} + (-1)^{n/2}\rho_2^{m-n/2} \right)\left(\rho_1^{n/2} - (-1)^{n/2}\rho_2^{n/2} \right)}{\rho_1 - \rho_2} \\
&= \frac{\left(\rho_1^{(p-1)/2} + (-1)^{n/2}\rho_2^{(p-1)/2} \right)\left(\rho_1^{n/2} - (-1)^{n/2}\rho_2^{n/2} \right)}{\rho_1 - \rho_2}.
\end{aligned}
$$

As a byproduct, by setting $u_q = (\rho_1^q - \rho_2^q)/(\rho_1 - \rho_2)$ and $v_q = \rho_1^q + \rho_2^q$ for any $q \in \mathbb{N}$,

$$A = \begin{cases} v_{(p-1)/2}\, u_{n/2} & \text{if } n \text{ is a multiple of } 4, \\ u_{(p-1)/2}\, v_{n/2} & \text{if } n \text{ is a multiple of } 4 \text{ plus } 2. \end{cases} \tag{14}$$

We conclude the case $b = 1$ as follows.

Theorem 5 *Fix a positive and odd integer p and let ρ be a root of the equation $r^{p+1} - r^p - r - 1 = 0$ different from $\pm i$. Set $a = \rho - 1/\rho$ and let us introduce a sequence $(x_n)_{n \in \mathbb{N}}$ satisfying the recurrence relation $x_{n+2} = a x_{n+1} + x_n$ for any $n \in \mathbb{N}$ and $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ the sequences satisfying the same relation such that $u_0 = 0$ and $u_1 = 1$, $v_0 = 2$ and $v_1 = a$ respectively. For any positive and even integer n such that $(n + p - 1)$ is a multiple of 4, the following identity holds:*

$$S_n = \left(u_{(n-p+1)/2} + u_{(n+p-1)/2} \right) x_{(n+p-1)/2} = \begin{cases} v_{(p-1)/2}\, u_{n/2}\, x_{(n+p-1)/2} & \text{if } n \text{ is a multiple of } 4, \\ u_{(p-1)/2}\, v_{n/2}\, x_{(n+p-1)/2} & \text{if } n \text{ is a multiple of } 4 \text{ plus } 2. \end{cases}$$

Examples 2 • *For $p = 1$, the corresponding equation reads $r^2 = 2r + 1$ whose solutions are $\left(1 + \sqrt{2}\right)$ and $\left(1 - \sqrt{2}\right)$. Hence $a = 2$, the recurrence relation for the sequence $(x_n)_{n \in \mathbb{N}}$ writes $x_{n+2} = 2x_{n+1} + x_n$ and the proportionality identity reads, for any integer n which is a multiple of 4,*

$$S_n = 2\, u_{n/2}\, x_{n/2}.$$

As an example of a possible trick, for $n = 12$, choose two numbers x_0 and x_1 and compute the 10 next numbers following the recurrence relation $x_{n+2} = 2x_{n+1} + x_n$:

$$x_2 = x_0 + 2x_1, \ x_3 = 2x_0 + 5x_1, \ x_4 = 5x_0 + 12x_1, \ x_5 = 12x_0 + 29x_1,$$
$$x_6 = 29x_0 + 70x_1, \ x_7 = 70x_0 + 169x_1, \ x_8 = 169x_0 + 408x_1,$$
$$x_9 = 408x_0 + 985x_1, \ x_{10} = 985x_0 + 2378x_1, \ x_{11} = 2378x_0 + 5741x_1.$$

The strategic number u_6 is obtained from x_6 by substituting therein the values 0 and 1 into x_0 and x_1: $u_6 = 70$. Evaluate now the sum of all these numbers $S = x_0 + x_1 + \cdots + x_{11}$. You find that

$$S = 4060x_0 + 9800x_1 = 140(29x_0 + 70x_1) = 140x_6 = 2u_6x_6.$$

- For $p = 3$, the corresponding equation writes $r^4 - r^3 - r - 1 = 0$ whose solutions are i, $-i$, φ_1 and φ_2. The roots φ_1 and φ_2 lead to $a = 1$ and we retrieve the Fibonacci sequence. The roots i and $-i$ are excluded from our analysis. The sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are the Fibonacci and Lucas sequences $(\mathcal{F}_n)_{n\in\mathbb{N}}$ and $(\mathcal{L}_n)_{n\in\mathbb{N}}$ respectively, and, for any integer n which is a multiple of 4 plus 2,

$$S_n = (\mathcal{F}_{n/2+1} + \mathcal{F}_{n/2-1})\, x_{n/2+1} = \mathcal{L}_{n/2}\, x_{n/2+1}.$$

- For $p = 5$, the corresponding equation writes $r^6 - r^5 - r - 1 = 0$ which admits two real solutions. Maple supplies the following estimates: -0.704 and 1.420 (we have $-1/1.420 \approx -0.704$). These both give the same possible value of a: $a \approx 0.715$; the recurrence relation for the corresponding sequence $(x_n)_{n\in\mathbb{N}}$ reads $x_{n+2} = ax_{n+1} + x_n$, and the proportionality identity reads, for any n which is a multiple of 4,

$$S_n = (u_{n/2+2} + u_{n/2-2})x_{n/2+2} = (a^2 + 2)u_{n/2}x_{n/2+2}.$$

We postpone to the appendix additional information about this example. In particular, we check this formula for $n = 8$.

In the table below, we collect estimates for certain coefficients a related to small values of the odd integer p supplied by Maple. The number ρ is the positive root of the polynomial $r^{p+1} - r^p - r - 1$, $a = \rho - 1/\rho$ determines the sequence for which a proportionality identity holds, and A is the proportionality coefficient.

p	ρ	a	m	A
1	2.414	2	$n/2$	$2\,u_{n/2}$
3	1.618	1	$n/2+1$	$v_{n/2}$
5	1.420	0.715	$n/2+2$	$(a^2+2)u_{n/2}$
7	1.325	0.570	$n/2+3$	$(a^2+1)v_{n/2}$
9	1.268	0.479	$n/2+4$	$(a^4+4a^2+2)u_{n/2}$
11	1.230	0.416	$n/2+5$	$(a^4+3a^2+1)v_{n/2}$
13	1.202	0.370	$n/2+6$	$(a^6+6a^4+9a^2+2)u_{n/2}$
15	1.181	0.334	$n/2+7$	$(a^6+5a^4+6a^2+1)v_{n/2}$
17	1.164	0.305	$n/2+8$	$(a^8+8a^6+20a^4+16a^2+2)u_{n/2}$
19	1.150	0.281	$n/2+9$	$(a^8+7a^6+15a^4+10a^2+1)v_{n/2}$

3.2   Solving the problem (continued)

The foregoing analysis suggests considering only the family of sequences $(x_n)_{n\in\mathbb{N}}$ following one of the recurrence relations $x_{n+2} = ax_{n+1} - x_n$ or $x_{n+2} = ax_{n+1} + x_n$ for any $n \in \mathbb{N}$, where $a \in \mathbb{C}$ is a fixed parameter. Actually, we essentially restrict ourselves to the first relation since the second one only concerns certain specific values of a. We briefly examine the second one at the end of this section.

For the sequences satisfying $x_{n+2} = ax_{n+1} - x_n$, we evaluate in this part the coefficient A given by (11) by means of the parameter a. We guess that the reader has certainly recognized the recurrence relation of the famous Chebyshev polynomials. We make clear this connection in Section 4. We propose here two representations for A.

As usual, $(u_n)_{n\in\mathbb{N}}$ denotes the sequence satisfying the same relation as $(x_n)_{n\in\mathbb{N}}$ with initial conditions $u_0 = 0$ and $u_1 = 1$. We recall the expression of u_n: $u_n = (r_1^n - r_2^n)/(r_1 - r_2)$ where r_1 and r_2 are the roots of $r^2 = ar - 1$, as well as that of A: $A = u_{(n+1)/2} + u_{(n-1)/2}$.

First approach

A first way of computing A consists in using generating functions. The generating function of $(u_n)_{n\in\mathbb{N}}$ is defined, for any complex number z such that $|z| < \min(|r_1|, |r_2|)$, by $G(z) = \sum_{n=0}^{\infty} u_n z^n$. It satisfies the following equation:

$$G(z) = u_0 + u_1 z + \sum_{n=2}^{\infty}(au_{n-1} - u_{n-2})z^n = z + az\sum_{n=2}^{\infty} u_{n-1}z^{n-1} - z^2\sum_{n=2}^{\infty} u_{n-2}z^{n-2}$$

$$= z + az\sum_{n=1}^{\infty} u_n z^n - z^2\sum_{n=0}^{\infty} u_n z^n = z + (az - z^2)G(z).$$

We immediately extract

$$G(z) = \frac{z}{1 - (az - z^2)}.$$

Now, we expand $G(z)$ into a power series by using the elementary identity $1/(1-\zeta) = \sum_{\ell=0}^{\infty}\zeta^\ell$:

$$G(z) = \sum_{\ell=0}^{\infty} z(az - z^2)^\ell = \sum_{\ell=0}^{\infty}\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}a^{\ell-k}z^{k+\ell+1} = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{\lfloor(n-1)/2\rfloor}(-1)^k\binom{n-k-1}{k}a^{n-2k-1}\right)z^n.$$

By identification, we derive the following representation of u_n:

$$u_n = \sum_{k=0}^{\lfloor(n-1)/2\rfloor}(-1)^k\binom{n-k-1}{k}a^{n-2k-1}.$$

As a result, with $m = (n-1)/2$,

$$A = u_{m+1} + u_m = \sum_{k=0}^{\lfloor m/2\rfloor}(-1)^k\left[\binom{m-k-1}{k}a^{m-2k-1} + \binom{m-k}{k}a^{m-2k}\right]. \tag{15}$$

In Formula (15), we adopt the convention that $\binom{\ell}{k} = 0$ if $k > \ell$. We observe that the coefficient A is a polynomial of the parameter a of the sequence $(x_n)_{n\in\mathbb{N}}$. This fact is actually not at all surprising and we detail this in Section 4.1.

Very similar computations would lead, for the sequence $(u_n)_{n\in\mathbb{N}}$ satisfying $u_0 = 0$, $u_1 = 1$ and the recurrence relation $u_{n+2} = au_{n+1} + u_n$, to

$$u_n = \sum_{k=0}^{\lfloor(n-1)/2\rfloor}\binom{n-k-1}{k}a^{n-2k-1}.$$

Second approach

A second expression for A can be found. For this, we examine several cases according to the values of a when $a \in \mathbb{R}\backslash\{-2, 2\}$ by using the explicit expressions of the roots of $r^2 = ar - 1$.

- If $|a| < 2$, we can write $a = 2\cos\theta$ for a certain $\theta \in (0, \pi)$ and the roots of $r^2 = ar - 1$ are $e^{i\theta}$ and $e^{-i\theta}$. Then $u_n = \sin(n\theta)/\sin\theta$ and

$$A = \frac{\sin((n+1)\theta/2) + \sin((n-1)\theta/2)}{\sin\theta} = \frac{\sin(n\theta/2)}{\sin(\theta/2)}.$$

- If $a > 2$, we can write $a = 2\cosh\theta$ for a certain $\theta > 0$ and the roots of $r^2 = ar - 1$ are e^θ and $e^{-\theta}$. Then $u_n = \sinh(n\theta)/\sinh\theta$ and

$$A = \frac{\sinh(n\theta/2)}{\sinh(\theta/2)}.$$

- If $a < -2$, we can write $a = -2\cosh\theta$ for a certain $\theta > 0$ and the roots of $r^2 = ar - 1$ are $-e^{\theta}$ and $-e^{-\theta}$. Then $u_n = (-1)^{n-1}\sinh(n\theta)/\sinh\theta$ and

$$A = (-1)^{(n-1)/2}\frac{\cosh(n\theta/2)}{\cosh(\theta/2)}.$$

Examples 3
- *For $a = 1$, the corresponding value of the angle θ is $\pi/3$. In this case, we have successively $x_2 = x_1 - x_0$, $x_3 = -x_0$, $x_4 = -x_1$, $x_5 = x_0 - x_1 = -x_2$, $x_6 = x_0$. We observe that the sequence $(x_n)_{n\in\mathbb{N}}$ is periodic: $x_{t+6} = x_t$ for any $t \in \mathbb{N}$. Next, the successive values of S_n are $S_0 = 0$, $S_1 = x_0$, $S_2 = x_0 + x_1$, $S_3 = 2x_1$, $S_4 = 2x_1 - x_0$, $S_5 = x_1 - x_0$, $S_6 = 0$. The sequence $(S_n)_{n\in\mathbb{N}}$ is also periodic: $S_{t+6} = S_t$. We can see that, for any $s, t \in \mathbb{N}$, $S_{6t+1} = x_0 = x_{6s}$, $S_{6t+3} = 2x_1 = 2x_{6s+1}$ and $S_{6t+5} = x_2 = x_{6s+2}$.*

- *For $a = -1$, the corresponding value of θ is $2\pi/3$. In this case, we have $x_2 = -x_0 - x_1$, $x_3 = x_0$ and $S_0 = 0$, $S_1 = x_0$, $S_2 = x_0 + x_1 = -x_2$, $S_3 = 0$. Thus, the sequences $(x_n)_{n\in\mathbb{N}}$ and $(S_n)_{n\in\mathbb{N}}$ are periodic: $x_{t+3} = x_t$ and $S_{t+3} = S_t$ for any $t \in \mathbb{N}$. We see that $S_{3t+1} = x_{3s}$ and $S_{3t+2} = -x_{3s+2}$.*

- *For $a = 0$, the corresponding value of θ is $\pi/2$. In this case, we have $x_2 = -x_0$, $x_3 = -x_1$, $x_4 = -x_2 = x_0$ and $S_0 = 0$, $S_1 = x_0$, $S_2 = x_0 + x_1$, $S_3 = x_1$, $S_4 = 0$. In this case again, the sequences are periodic and $S_{4t+1} = x_0 = x_{4s}$ and $S_{4t+3} = x_1 = x_{4s+1}$ for any $s, t \in \mathbb{N}$.*

Remark 4 *The sequence $(x_n)_{n\in\mathbb{N}}$ is periodic if and only if the angles θ and π are commensurable. In this case, in view of (7), $(S_n)_{n\in\mathbb{N}}$ is clearly periodic.*

We end up this section by examining briefly the recurrence relation $x_{n+2} = ax_{n+1} + x_n$. We recall that we found a solution to our problem only for certain values of a (see Theorem 5). We can write $a = 2\sinh\theta$ for a certain $\theta \in \mathbb{R}$ and the roots of the equation $r^2 = ar + 1$ are e^{θ} and $-e^{-\theta}$. Then $u_n = \sinh(n\theta)/\cosh\theta$ if n is even and the expression (13) of the coefficient A becomes

$$A = \frac{\sinh((n+p-1)\theta/2) + \sinh((n-p+1)\theta/2)}{\cosh\theta} = 2\frac{\sinh(n\theta/2)\cosh((p-1)\theta/2)}{\cosh\theta}$$

where n and p are positive integers such that n is even, p is odd and $(n+p-1)$ is a multiple of 4.

4 Link with Fibonacci, Lucas and Chebyshev polynomials

In this part, we make clear the link between the sequences defined by the recurrence relations $x_{n+2} = ax_{n+1} \pm x_n$ that emerged in the foregoing section and the famous Fibonacci, Lucas and Chebyshev polynomials.

4.1 Link with Chebyshev polynomials

We recall the definition of the families of Chebyshev polynomials $(T_n)_{n\in\mathbb{N}}$ (first kind) and $(U_n)_{n\in\mathbb{N}}$ (second kind) through the expansion of the trigonometric quantities $\cos(n\theta)$ and $\sin(n\theta)$ (or hyperbolic quantities $\cosh(n\theta)$ and $\sinh(n\theta)$) as polynomials of $\cos\theta$ (or $\cosh(n\theta)$):

$$\cos(n\theta) = T_n(\cos\theta) \text{ and } \sin(n\theta) = \sin(\theta)U_{n-1}(\cos\theta).$$

We refer to [7], [8] and [12]. They are characterized by the initial conditions $T_0(x) = 1$, $T_1(x) = x$, $U_0(x) = 1$, $U_1(x) = 2x$ for any $x \in \mathbb{C}$ and the relationships, for any $n \in \mathbb{N}$ and $x \in \mathbb{C}$,

$$T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x) \text{ and } U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x).$$

Additionally, we put $U_{-1}(x) = 0$. Regarding the sequence we were dealing with in Section 3.2, it is convenient to introduce $\tilde{T}_n(x) = 2T_n(x/2)$ and $\tilde{U}_n(x) = U_{n-1}(x/2)$. In the literature, $\tilde{T}_n$ and $\tilde{U}_n$ are usually denoted by C_n and S_{n-1} respectively. We have the initial conditions $\tilde{T}_0(x) = 2$, $\tilde{T}_1(x) = x$ and $\tilde{U}_0(x) = 0$, $\tilde{U}_1(x) = 1$, together with the recurrence relations

$$\tilde{T}_{n+2}(x) = x\tilde{T}_{n+1}(x) - \tilde{T}_n(x), \quad \tilde{U}_{n+2}(x) = x\tilde{U}_{n+1}(x) - \tilde{U}_n(x).$$

The polynomials $\tilde{T}_n$ and $\tilde{U}_n$ obey the following identities:

$$\tilde{T}_n(2\cos\theta) = 2\cos(n\theta), \qquad\qquad \tilde{U}_n(2\cos\theta) = \frac{\sin(n\theta)}{\sin\theta},$$

$$\tilde{T}_n(2\cosh\theta) = 2\cosh(n\theta), \qquad\qquad \tilde{U}_n(2\cosh\theta) = \frac{\sinh(n\theta)}{\sinh\theta},$$

$$\tilde{T}_n(-2\cosh\theta) = 2(-1)^n\cosh(n\theta), \quad \tilde{U}_n(-2\cosh\theta) = (-1)^{n-1}\frac{\sinh(n\theta)}{\sinh\theta},$$

and $\tilde{U}_n$ admits the following explicit expression, for any $x \in \mathbb{C}$,

$$\tilde{U}_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} (-1)^k \binom{n-k-1}{k} x^{n-2k-1}.$$

As a byproduct, the sequence $(u_n)_{n\in\mathbb{N}}$ introduced in Theorem 4 is directly related to the family of polynomials $(\tilde{U}_n)_{n\in\mathbb{N}}$: in fact, $u_n = \tilde{U}_n(a)$. Then $A = \tilde{U}_{m+1}(a) + \tilde{U}_m(a)$ with $m = (n-1)/2$. This formula holds for any $a \in \mathbb{C}$. The complement of Theorem 4 is the following statement.

Theorem 6 *Fix a complex number a and an odd positive integer n. The factor A in the relationship $S_n = Ax_{(n-1)/2}$ related to any sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the recurrence relation $x_{n+2} = ax_{n+1} - x_n$ is explicitly given by*

$$A = \sum_{k=0}^{\lfloor (n-1)/4 \rfloor} (-1)^k \left[\binom{(n-3)/2-k}{k} a^{(n-3)/2-2k} + \binom{(n-1)/2-k}{k} a^{(n-1)/2-2k} \right].$$

We then retrieve (15). We point out that when a is an integer, so is A. The reader can compare this expression of A with the various trigonometric and hyperbolic representations obtained in Section 3.2.

Actually, the relationship $S_n = Ax_m$ is nothing but the translation of a well-known identity concerning the sum of trigonometric (or hyperbolic) functions. Indeed, recall that, e.g.,

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \cos(n\theta/2), \quad \sum_{k=0}^{n} \sin(k\theta) = \frac{\sin((n+1)\theta/2)}{\sin(\theta/2)} \sin(n\theta/2).$$

By means of Cheybyshev polynomials, for $n = 2m+1$, these two formulas can be rewritten as

$$\sum_{k=0}^{n-1} \tilde{T}_k(2\cos\theta) = \frac{\sin((m+1/2)\theta)}{\sin(\theta/2)} 2\cos(m\theta) = \frac{\sin(m\theta) + \sin((m+1)\theta)}{\sin\theta} 2\cos(m\theta)$$

$$= [\tilde{U}_m(2\cos\theta) + \tilde{U}_{m+1}(2\cos\theta)] \tilde{T}_m(2\cos\theta), \tag{16}$$

$$\sum_{k=0}^{n-1} \tilde{U}_k(2\cos\theta) = \frac{\sin((m+1/2)\theta)}{\sin(\theta/2)} \frac{\sin(m\theta)}{\sin\theta} = \frac{\sin(m\theta) + \sin((m+1)\theta)}{\sin\theta} \frac{\sin(m\theta)}{\sin\theta}$$

$$= [\tilde{U}_m(2\cos\theta) + \tilde{U}_{m+1}(2\cos\theta)] \tilde{U}_m(2\cos\theta). \tag{17}$$

Formulas (16) and (17) can be simply written, with $A = \tilde{U}_{m+1}(a) + \tilde{U}_m(a)$, as

$$\sum_{k=0}^{n-1} \tilde{T}_k(a) = A\,\tilde{T}_m(a) \ \text{ and } \ \sum_{k=0}^{n-1} \tilde{U}_k(a) = A\,\tilde{U}_m(a). \tag{18}$$

Since any sequence $(x_n)_{n\in\mathbb{N}}$ satisfying the recurrence $x_{n+2} = ax_{n+1} - x_n$ is a linear combination of the sequences $(\tilde{T}_n(a))_{n\in\mathbb{N}}$ and $(\tilde{U}_n(a))_{n\in\mathbb{N}}$, the relationships (18) can be extended to the sequence $(x_n)_{n\in\mathbb{N}}$, that is

$$\sum_{k=0}^{n-1} x_k = Ax_m.$$

4.2   Link with Fibonacci and Lucas polynomials

We recall the definition of the families of Fibonacci polynomials $(F_n)_{n\in\mathbb{N}}$ and Lucas polynomials $(L_n)_{n\in\mathbb{N}}$ through the expansion of the hyperbolic quantities $\cosh(n\theta)$ and $\sinh(n\theta)$ as polynomials of $\sinh\theta$ (see [10], [14]):

$$F_n(2\sinh\theta) = \begin{cases} \dfrac{\sinh(n\theta)}{\cosh\theta} & \text{if } n \text{ is even,} \\ \dfrac{\cosh(n\theta)}{\cosh\theta} & \text{if } n \text{ is odd,} \end{cases} \qquad L_n(2\sinh\theta) = \begin{cases} 2\cosh(n\theta) & \text{if } n \text{ is even,} \\ 2\sinh(n\theta) & \text{if } n \text{ is odd.} \end{cases}$$

They are characterized by the initial conditions $F_0(x) = 0$, $F_1(x) = 1$, $L_0(x) = 2$, $L_1(x) = x$ and the relationships, for any $n \in \mathbb{N}$,

$$F_{n+2}(x) = xF_{n+1}(x) + F_n(x) \ \text{ and } \ L_{n+2}(x) = xL_{n+1}(x) + L_n(x).$$

The polynomials F_n and L_n admit the following explicit expansions:

$$F_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-k-1}{k} x^{n-2k-1}, \qquad L_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k}\binom{n-k}{k} x^{n-2k}.$$

The sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ introduced in Theorem 5 are directly related to the families of polynomials $(F_n)_{n\in\mathbb{N}}$ and $(L_n)_{n\in\mathbb{N}}$: indeed, $u_n = F_n(a)$ and $v_n = L_n(a)$. The two expressions of the coefficient A we have obtained in (13) and (14) can be rewritten as

$$A = F_{(n+p-1)/2}(a) + F_{(n-p+1)/2}(a) = \begin{cases} L_{(p-1)/2}(a)\, F_{n/2}(a) & \text{if } n \text{ is a multiple of 4,} \\ F_{(p-1)/2}(a)\, L_{n/2}(a) & \text{if } n \text{ is a multiple of 4 plus 2.} \end{cases}$$

We recall that here n is an even integer and p an odd integer such that $(n+p-1)$ is a multiple of 4. Hence $n/2$ and $(p-1)/2$ have the same parity. The first above equality explicitly yields

$$A = \sum_{k=0}^{\lfloor (n+p-3)/4 \rfloor} \binom{(n+p-3)/2-k}{k} a^{(n+p-3)/2-2k} + \sum_{k=0}^{\lfloor (n-p-1)/4 \rfloor} \binom{(n-p-1)/2-k}{k} a^{(n-p-1)/2-2k}.$$

This formula holds only for those $a \in \mathbb{C}$ which can be written as $a = \rho - 1/\rho$ where ρ satisfies the equation $\rho^{p+1} - \rho^p - \rho - 1 = 0$.

Remark 5 *The equality between both expressions (13) and (14) is actually the rephrasing of an elementary identity concerning the hyperbolic functions. Indeed, let s and t be two integers with the same parity. Then,*

$$F_{s+t}(2\sinh\theta) + F_{s-t}(2\sinh\theta) = \frac{\sinh((s+t)\theta) + \sinh((s-t)\theta)}{\cosh\theta}$$
$$= 2\cosh(t\,\theta)\frac{\sinh(s\,\theta)}{\cosh\theta} = 2\sinh(s\,\theta)\frac{\cosh(t\,\theta)}{\cosh\theta}$$
$$= \begin{cases} L_t(2\sinh\theta)F_s(2\sinh\theta) & \text{if } s \text{ and } t \text{ are even,} \\ L_s(2\sinh\theta)F_t(2\sinh\theta) & \text{if } s \text{ and } t \text{ are odd.} \end{cases}$$

More concisely,

$$F_{s+t}(a) + F_{s-t}(a) = \begin{cases} L_t(a)\, F_s(a) & \text{if } s \text{ and } t \text{ are even,} \\ F_t(a)\, L_s(a) & \text{if } s \text{ and } t \text{ are odd.} \end{cases}$$

Choosing $s = n/2$ and $t = (p-1)/2$ in the foregoing equality confirms the equality of (13) and (14).

Remark 6 *It is well-known that the Fibonacci and Lucas polynomials are simply related to the Chebyshev polynomials by*

$$\tilde{T}_n(ix) = i^n L_n(x) \ \text{ and } \ \tilde{U}_n(ix) = i^{n-1}F_n(x).$$

Despite this connection, because of the factor i^n, there is no simple link between, e.g., the sums $\sum_{k=0}^{n-1}\tilde{U}_k$ and $\sum_{k=0}^{n-1} F_k$. Hence, specific studies were necessary for the cases $b = 1$ and $b = -1$.

5 Conclusion

In this article, we have considered sequences $(x_n)_{n\in\mathbb{N}}$ defined by two initial conditions x_0 and x_1 and a second order linear recurrence relation: $x_{n+2} = ax_{n+1} + bx_n$ for any $n \in \mathbb{N}$. In several cases, for any $n \in \mathbb{N}$, we have been able to get an integer m and a number A such that $\sum_{k=0}^{n-1} x_k = Ax_m$. More precisely, such relationship is tractable for any $a \in \mathbb{C}$ when $b = -1$, and for certain specific values of a when $b = 1$. The coefficient A can be expressed in terms of Chebyshev polynomials when $b = 1$ and in terms of Fibonacci polynomials when $b = -1$.

In the table below, we sum up the values of m and A obtained throughout this study.

recurrence	integer m	coefficient A
$x_{n+2} = x_{n+1} + x_n$	$n/2 + 1$ (n multiple of 4 plus 2)	$\mathcal{L}_{n/2}$
$x_{n+2} = -x_{n+1} + x_n$	$n/2 - 2$ (n multiple of 4 plus 2)	$\mathcal{L}_{n/2}$
$x_{n+2} = 2x_{n+1} + x_n$	$n/2$ (n multiple of 4)	$2\,F_{n/2}(2)$
$x_{n+2} = 2x_{n+1} - x_n$	$(n-1)/2$ (odd n)	n
$x_{n+2} = -2x_{n+1} - x_n$	$(n-1)/2$ (odd n)	$(-1)^{(n-1)/2}$
$x_{n+2} = ax_{n+1} - x_n$	$(n-1)/2$ (odd n)	$\tilde{U}_{(n+1)/2}(a) + \tilde{U}_{(n-1)/2}(a)$
$x_{n+2} = ax_{n+1} + x_n$ for certain values of $a \in \mathbb{C}$	$(n+p-1)/2$ ($n+p-1$ multiple of 4)	$F_{(n+p-1)/2}(a)$ $+ F_{(n-p+1)/2}(a)$

A Appendix

A.1 An example

In this appendix, we go back to the example of the recurrence $x_{n+2} = ax_{n+1} + x_n$ with $a = \rho - 1/\rho$ where ρ is the positive root of $r^6 - r^5 - r - 1 = 0$; see the case $p = 5$ of Examples 2. Starting from $\rho^6 = \rho^5 + \rho + 1$, we get

$$a = \frac{\rho^2 - 1}{\rho} = \frac{\rho^3 - \rho}{\rho^2},$$

$$a^2 = \frac{\rho^4 - 2\rho^2 + 1}{\rho^2},$$

$$a^3 = \frac{\rho^6 - 3\rho^4 + 3\rho^2 - 1}{\rho^3} = \frac{\rho^4 - 3\rho^3 + 3\rho + 1}{\rho^2}.$$

We see that a solves the following equation of degree 3:

$$a^3 - a^2 + 3a - 2 = 0.$$

Appealing to the famous Cardano's formula, we can extract the exact real value of a:

$$a = \frac{1}{6}\left(\sqrt[3]{116 + 12\sqrt{321}} - \sqrt[3]{-116 + 12\sqrt{321}}\right) + \frac{1}{3}.$$

As an example of the formulas $S_n = (u_{n/2+2} + u_{n/2-2})x_{n/2+2} = (a^2 + 2)u_{n/2}x_{n/2+2}$, we examine the case $n = 8$. We begin by writing the first terms of the sequence $(x_n)_{n\in\mathbb{N}}$:

$$x_2 = x_0 + ax_1,$$
$$x_3 = ax_0 + (a^2 + 1)x_1,$$
$$x_4 = (a^2 + 1)x_0 + (a^3 + 2a)x_1,$$
$$x_5 = (a^3 + 2a)x_0 + (a^4 + 3a^2 + 1)x_1,$$
$$x_6 = (a^4 + 3a^2 + 1)x_0 + (a^5 + 4a^3 + 3a)x_1,$$
$$x_7 = (a^5 + 4a^3 + 3a)x_0 + (a^6 + 5a^4 + 6a^2 + 1)x_1.$$

For $x_0 = 0$ and $x_1 = 1$, we derive $u_2 = a$, $u_4 = a^3 + 2a$ and $u_6 = a^5 + 4a^3 + 3a$. We observe that $u_2 + u_6 = a^5 + 4a^3 + 4a = (a^2 + 2)u_4$. The sum of all numbers from x_0 to x_7 is given by

$$S_8 = (a^5 + a^4 + 5a^3 + 4a^2 + 6a + 4)x_0 + (a^6 + a^5 + 6a^4 + 5a^3 + 10a^2 + 6a + 4)x_1. \qquad \text{(A.1)}$$

On the other hand,

$$(u_2 + u_6)x_6 = (a^5 + 4a^3 + 4a)[(a^4 + 3a^2 + 1)x_0 + (a^5 + 4a^3 + 3a)x_1]$$
$$= (a^9 + 7a^7 + 17a^5 + 16a^3 + 4a)x_0 + (a^{10} + 8a^8 + 23a^6 + 28a^4 + 12a^2)x_1. \qquad \text{(A.2)}$$

Our aim is to check that (A.1) and (A.2) coincide. Using $a^3 = a^2 - 3a + 2$, we successively obtain

$$a^4 = a^3 - 3a^2 + 2a = -2a^2 - a + 2,$$
$$a^5 = -2a^3 - a^2 + 2a = -3a^2 + 8a - 4,$$
$$a^6 = -3a^3 + 8a^2 - 4a = 5a^2 + 5a - 6,$$
$$a^7 = 5a^3 + 5a^2 - 6a = 10a^2 - 21a + 10,$$
$$a^8 = 10a^3 - 21a^2 + 10a = -11a^2 - 20a + 20,$$
$$a^9 = -11a^3 - 20a^2 + 20a = -31a^2 + 53a - 22,$$
$$a^{10} = -31a^3 + 53a^2 - 22a = 22a^2 + 71a - 62.$$

We easily get

$$u_2 + u_6 = a^5 + 4a^3 + 4a = a^2 + 4,$$
$$a^5 + a^4 + 5a^3 + 4a^2 + 6a + 4 = a^9 + 7a^7 + 17a^5 + 16a^3 + 4a = 4a^2 - 2a + 12,$$
$$a^6 + a^5 + 6a^4 + 5a^3 + 10a^2 + 6a + 4 = a^{10} + 8a^8 + 23a^6 + 28a^4 + 12a^2 = 5a^2 - 2a + 16.$$

As a result, we just checked that

$$S_8 = (u_2 + u_6)x_6 = (4a^2 - 2a + 12)x_0 + (5a^2 - 2a + 16)x_1$$

and we simply have the relationship

$$S_8 = (a^2 + 4)x_6.$$

A.2 An algebraic equation

In this second appendix, we still consider the example of the recurrence relation $x_{n+2} = ax_{n+1} + x_n$ treated in Theorem 5. Therein, $a = \rho - 1/\rho$ where ρ is a root of $r^{p+1} - r^p - r - 1 = 0$, p being an odd integer. Of course, since ρ is an algebraic number, so is a. Our aim is to write out an algebraic equation satisfied by a. We distinguish the two cases when p is a multiple of 4 plus or minus 1: $p = 4q + 1$ or $p = 4q - 1$ say.

The case $p = 4q + 1$

Suppose that p is of the form $p = 4q + 1$ for a positive integer q. We have $\rho^{4q+2} - \rho^{4q+1} - \rho - 1 = 0$. The equation we have obtained for a is an equation of degree $2q + 1$. Our method is heuristic: we search an equation of the form

$$a^{2q+1} = \alpha_{2q}a^{2q} + \alpha_{2q-1}a^{2q-1} + \cdots + \alpha_1 a + \alpha_0 = \sum_{j=0}^{2q} \alpha_j a^j$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{2q}$ are some integers that we are going to determine.

Let us evaluate the successive powers of a: for any positive integer j,

$$a^j = \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j}.$$

In particular, for $j = 2q + 1$,

$$a^{2q+1} = -\sum_{k=0}^{2q+1} (-1)^k \binom{2q+1}{k} \rho^{2k-2q-1} = \rho^{2q+1} - \rho^{-2q-1} - \sum_{k=1}^{2q} (-1)^k \binom{2q+1}{k} \rho^{2k-2q-1}.$$

Since $\rho^{4q+2} - \rho^{4q+1} - \rho - 1 = 0$, we have $\rho^{2q+1} - \rho^{-2q-1} = \rho^{2q} + \rho^{-2q}$. Then,

$$a^{2q+1} = \rho^{2q} + \rho^{-2q} - \sum_{k=1}^{2q} (-1)^k \binom{2q+1}{k} \rho^{2k-2q-1}.$$

The equation we are looking for writes as

$$\sum_{j=0}^{2q} \alpha_j \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j} + \sum_{k=1}^{2q} (-1)^k \binom{2q+1}{k} \rho^{2k-2q-1} - (\rho^{2q} + \rho^{-2q}) = 0. \tag{A.3}$$

The double sum of the left-hand side of (A.3) can be written as follows:

$$\sum_{j=0}^{2q} \alpha_j \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j} = \sum_{i=-2q}^{2q} \left[\sum_{\substack{j,k \in \mathbb{N}: \, 0 \le k \le j \le 2q \\ 2k-j=i}} (-1)^{j+k} \binom{j}{k} \alpha_j \right] \rho^i$$

$$= \sum_{i=-2q}^{2q} \left[\sum_{\substack{j \in \mathbb{N}: \, |i| \le j \le 2q, \\ i,j \text{ have the same parity}}} (-1)^{(i+3j)/2} \binom{j}{(i+j)/2} \alpha_j \right] \rho^i$$

$$= \alpha_{2q}(\rho^{2q} + \rho^{-2q}) + \sum_{\substack{i \in \mathcal{E}: \\ |i| \le 2q-1}} \left[\sum_{\substack{j \in \mathcal{E}: \\ |i| \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] \rho^i$$

$$+ \sum_{\substack{i \in \mathcal{O}: \\ |i| \le 2q-1}} \left[\sum_{\substack{j \in \mathcal{O}: \\ |i| \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] \rho^i. \tag{A.4}$$

In the last equality, $\mathcal{E}$ and $\mathcal{O}$ denote the sets of even and odd integers respectively. By using symmetry properties of the binomial coefficients, it is enough in (A.4) to sum over positive integers:

$$\sum_{j=0}^{2q} \alpha_j \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j} = \alpha_{2q}(\rho^{2q} + \rho^{-2q}) + \sum_{\substack{j \in \mathcal{E}: \\ 0 \le j \le 2q}} (-1)^{j/2} \binom{j}{j/2} \alpha_j$$

$$+ \sum_{\substack{i \in \mathcal{E}: \\ 1 \le i \le 2q-1}} \left[\sum_{\substack{j \in \mathcal{E}: \\ i \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] (\rho^i + \rho^{-i})$$

$$+ \sum_{\substack{i \in \mathcal{O}: \\ 1 \le i \le 2q-1}} \left[\sum_{\substack{j \in \mathcal{O}: \\ i \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] (\rho^i - \rho^{-i}). \tag{A.5}$$

On the other hand,

$$\sum_{k=1}^{2q} (-1)^k \binom{2q+1}{k} \rho^{2k-2q-1} = \sum_{\substack{i \in \mathcal{O}: \\ 1 \le i \le 2q-1}} (-1)^{(i+1)/2+q} \binom{2q+1}{(i+1)/2+q} (\rho^i - \rho^{-i}). \tag{A.6}$$

Plugging (A.5) and (A.6) into (A.3) yields

$$(\alpha_{2q} - 1)(\rho^{2q} + \rho^{-2q}) + \sum_{\substack{j \in \mathcal{E}: \\ 0 \le j \le 2q}} (-1)^{j/2} \binom{j}{j/2} \alpha_j + \sum_{\substack{i \in \mathcal{E}: \\ 1 \le i \le 2q-1}} \left[\sum_{\substack{j \in \mathcal{E}: \\ i \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] (\rho^i + \rho^{-i})$$

$$+ \sum_{\substack{i \in \mathcal{O}: \\ 1 \le i \le 2q-1}} \left[(-1)^{(i+1)/2+q} \binom{2q+1}{(i+1)/2+q} + \sum_{\substack{j \in \mathcal{O}: \\ i \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] (\rho^i - \rho^{-i}) = 0.$$

$$\tag{A.7}$$

With obvious notations, Equation (A.7) is clearly of the form

$$\sum_{i=0}^{2q} \beta_i (\rho^i + (-1)^i \rho^{-i}) = 0$$

which is satisfies as soon as the β_i's vanish (this is a sufficient condition which is not *a priori* necessary). Hence, (A.7) is satisfied when $\alpha_{2q} = 1$ and

$$\begin{cases} \displaystyle\sum_{\substack{j \in \mathcal{E}: \\ i \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j = 0 & \text{for } i \in \{0, 2, 4, \ldots, 2q-2\}, \\ \displaystyle\sum_{\substack{j \in \mathcal{O}: \\ i \le j \le 2q}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j = (-1)^{(i-1)/2+q} \binom{2q+1}{(i+1)/2+q} & \text{for } i \in \{1, 3, 5, \ldots, 2q-1\}. \end{cases}$$

We rewrite this system by changing the indices i and j into $2i$ and $2j$ in the first equation, $(2i+1)$ and $(2j+1)$ in the second one:

$$\begin{cases} \displaystyle\sum_{j=i}^{q-1} (-1)^j \binom{2j}{j-i} \alpha_{2j} = (-1)^{q+1} \binom{2q}{q-i} & \text{for } i \in \{0, 1, 2, \ldots, q-1\}, \\ \displaystyle\sum_{j=i}^{q-1} (-1)^j \binom{2j+1}{j-i} \alpha_{2j+1} = (-1)^q \binom{2q+1}{q-i} & \text{for } i \in \{0, 1, 2, \ldots, q-1\}. \end{cases} \tag{A.8}$$

Let us introduce the principal matrices related to this system:

$$T_1 = \left[(-1)^j \binom{2j}{j-i} \right]_{0 \le i,j \le q-1} \quad \text{and} \quad T_2 = \left[(-1)^j \binom{2j+1}{j-i} \right]_{0 \le i,j \le q-1}$$

as well as the matrices of the unknowns and those of the right-hand sides of (A.8):

$$A_1 = [\alpha_{2i}]_{0 \le i \le q-1}, \quad A_2 = [\alpha_{2i+1}]_{0 \le i \le q-1},$$

$$B_1 = (-1)^{q+1} \left[\binom{2q}{q-i} \right]_{0 \le i \le q-1}, \quad B_2 = (-1)^q \left[\binom{2q+1}{q-i} \right]_{0 \le i \le q-1}.$$

Recalling the convention that $\binom{\ell}{k} = 0$ if $k > \ell$, we notice that the matrices T_1 and T_2 are triangular. With these settings at hand, the system (A.8) simply reads

$$T_1 A_1 = B_1 \quad \text{and} \quad T_2 A_2 = B_2.$$

We need to compute the inverse of T_1 and T_2. By performing empirical computations with the help of Maple, we found that the entries of T_1^{-1} and $A_1 = T_1^{-1} B_1$ are respectively

$$(-1)^j \binom{2j-2i}{j-i} \binom{2j}{2i} / \binom{2j-1}{j-i} \quad \text{and} \quad \binom{2q}{2i} \binom{2q-2i}{q-i} / \binom{2q-1}{q-i}.$$

We found analogous expressions T_2^{-1} and $A_2 = T_2^{-1} B_2$. All of them can be simplified. We state the results in the Propositions A.1 and A.2.

Proposition A.1 *We have*

$$T_1^{-1} = \left[(-1)^j \frac{2j}{i+j} \binom{i+j}{2i} \right]_{0 \le i,j \le q-1},$$

$$T_2^{-1} = \left[(-1)^j \frac{2j+1}{i+j+1} \binom{i+j+1}{2i+1} \right]_{0 \le i,j \le q-1}.$$

In T_1^{-1} above, we adopt the convention that $(2j)/(i+j) = 1$ if $i = j = 0$.

PROOF.

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We only prove the formula related to T_1^{-1}, the proof of that related to T_2^{-1} being quite similar. We check that the product of upper triangular matrices

$$\left[(-1)^j\binom{2j}{i+j}\right]_{0\le i,j\le q-1} \times \left[(-1)^j\frac{2j}{i+j}\binom{i+j}{2i}\right]_{0\le i,j\le q-1} \tag{A.9}$$

exactly coincides with the unit matrix of type $q\times q$, that is, $T_1T_1^{-1}=I_q$. For this, we have to compute its upper entries: for $0\le i\le q-1$ and $1\le j\le q-1$ such that $i\le j$,

$$\sum_{k=0}^{q-1}(-1)^{j+k}\frac{2j}{k+j}\binom{2k}{k+i}\binom{k+j}{2k} = 2j\sum_{k=i}^{j}(-1)^{j+k}\frac{(k+j-1)!}{(j-k)!(k-i)!(k+i)!}$$

$$= (-1)^j\frac{2j}{(j-i)!}\sum_{k=i}^{j}(-1)^k\frac{(k+j-1)!}{(k+i)!}\binom{j-i}{k-i}$$

$$= (-1)^{i+j}\frac{2j}{(j-i)!}\sum_{k=0}^{j-i}(-1)^k\frac{(k+i+j-1)!}{(k+2i)!}\binom{j-i}{k}.$$

In order to evaluate the last sum, we use the lemma below: applying (A.10) to $\ell=2i+j$ and $m=j-i$ immediately yields, for $0\le i\le q-1$ and $1\le j\le q-1$ such that $i\le j$,

$$\sum_{k=0}^{q-1}(-1)^{j+k}\frac{2j}{k+j}\binom{2k}{k+i}\binom{k+j}{2k} = 0.$$

With the convention $(2j)/(k+j)=1$ for $j=k=0$, the foregoing sum is equal to 1. This proves that the product (A.9) coincides with the unit matrix. □

Lemma A.1 *For any positive integers ℓ and m such that $\ell\ge m$, we have*

$$\sum_{k=0}^{m}(-1)^k\frac{(k+\ell-1)!}{(k+\ell-m)!}\binom{m}{k} = 0. \tag{A.10}$$

PROOF.

By observing that $\dfrac{\mathrm{d}^{m-1}}{\mathrm{d}^{m-1}x}(x^{k+\ell-1}) = \dfrac{(k+\ell-1)!}{(k+\ell-m)!}x^{k+\ell-m}$, we get

$$\sum_{k=0}^{m}(-1)^k\frac{(k+\ell-1)!}{(k+\ell-m)!}\binom{m}{k}x^k = x^{m-\ell}\frac{\mathrm{d}^{m-1}}{\mathrm{d}^{m-1}x}\left(\sum_{k=0}^{m}(-1)^k\binom{m}{k}x^{k+\ell-1}\right) = x^{m-\ell}\frac{\mathrm{d}^{m-1}}{\mathrm{d}^{m-1}x}[(1-x)^m x^{\ell-1}].$$

It is clear that 1 is a root of $(1-x)^m x^{\ell-1}$ of order m, so that the $(m-1)$th derivative of $(1-x)^m x^{\ell-1}$ vanishes at $x=1$. This finishes the proof of (A.10). □

Proposition A.2 *We have*

$$A_1 = \left[\frac{2q}{q+i}\binom{q+i}{2i}\right]_{0\le i\le q-1}, \quad A_2 = -\left[\frac{2q+1}{q+i+1}\binom{q+i+1}{2i+1}\right]_{0\le i\le q-1}.$$

PROOF.

We only prove the formula related to A_1, the proof of that related to A_2 being quite similar. We have to compute the product $A_1 = T_1^{-1}B_1$:

$$\left[(-1)^j\frac{2j}{i+j}\binom{i+j}{2i}\right]_{0\le i,j\le q-1} \times \left[(-1)^{q+1}\binom{2q}{q-i}\right]_{0\le i\le q-1}.$$

The entries of this matrix are

$$\alpha_{2i} = \sum_{k=i}^{q-1}(-1)^{k+q+1}\frac{2k}{k+i}\binom{k+i}{2i}\binom{2q}{q-k}.$$

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We previously proved that the product (A.9) coincides with the unit matrix, that is $T_1 T_1^{-1} = I_q$. Of course, in this case, the product is commutative and we have also $T_1^{-1} T_1 = I_q$. This remark entails the following identity: for any integers i, j such that $0 \leq i \leq j \leq q - 1$,

$$\sum_{k=i}^{j} (-1)^{j+k} \frac{2k}{k+i} \binom{k+i}{2i} \binom{2j}{k+j} = \delta_{ij}.$$

Noticing that this formula actually does not depend on the number q, it holds true also for $j = q$, which gives, for $0 \leq i \leq q - 1$,

$$\sum_{k=i}^{q} (-1)^{k} \frac{2k}{k+i} \binom{k+i}{2i} \binom{2q}{k+q} = 0. \tag{A.11}$$

As a result, by isolating the last term of the previous sum, we finally get, for $0 \leq i \leq q - 1$,

$$\sum_{k=i}^{q-1} (-1)^{k+q+1} \frac{2k}{k+i} \binom{k+i}{2i} \binom{2q}{q-k} = \frac{2q}{q+i} \binom{q+i}{2i}.$$

This proves the formula obtained for A_1. $\qquad\square$

The conclusion of this analysis is the following equation satisfied by a.

Theorem A.1 *Let q be a positive integer and ρ be a root of the equation $r^{4q+2} = r^{4q+1} + r + 1$. The parameter $a = \rho - 1/\rho$ is an algebraic number; it solves the equation*

$$a^{2q+1} = \sum_{i=0}^{q} \frac{2q}{q+i} \binom{q+i}{2i} a^{2i} - \sum_{i=0}^{q-1} \frac{2q+1}{q+i+1} \binom{q+i+1}{2i+1} a^{2i+1}.$$

Examples 4 *For $q \in \{1, 2, 3, 4\}$ ($p \in \{5, 9, 13, 17\}$), the corresponding equation satisfied by a reads*

$$a^3 - a^2 + 3a - 2 = 0,$$
$$a^5 - a^4 + 5a^3 - 4a^2 + 5a - 2 = 0,$$
$$a^7 - a^6 + 7a^5 - 6a^4 + 14a^3 - 9a^2 + 7a - 2 = 0,$$
$$a^9 - a^8 + 9a^7 - 8a^6 + 27a^5 - 20a^4 + 30a^3 - 16a^2 + 9a - 2 = 0.$$

The case $p = 4q - 1$

Assume now that $p = 4q - 1$ for a positive integer q. We have in this case $\rho^{4q} - \rho^{4q-1} - \rho - 1 = 0$. The previous calculations cannot be carried out exactly in the same way and we must work with a slightly modified equation. In this situation, we observe that i and $-i$ are particular solutions; we also choose a different from $\pm 2i$ as required in Theorem 5. Thus, the polynomial $r^{4q} - r^{4q-1} - r - 1$ can be divided by $r^2 + 1$ and the quotient is

$$r^{4q-2} - r^{4q-3} - r^{4q-4} + r^{4q-5} + r^{4q-6} - \cdots + r^3 + r^2 - r - 1 = r^{4q-2} - \sum_{k=0}^{2q-2} (-1)^k (r^{2k} + r^{2k+1}).$$

The equation we have obtained for a is an equation of degree $2q - 1$. As previously, we look for an equation of the form

$$a^{2q-1} = \alpha_{2q-2} a^{2q-2} + \alpha_{2q-3} a^{2q-3} + \cdots + \alpha_1 a + \alpha_0 = \sum_{j=0}^{2q-2} \alpha_j a^j$$

where $\alpha_0, \alpha_1, \ldots, \alpha_{2q-2}$ are some integers that we are going to determine. The successive powers of a write, for any positive integer j, as

$$a^j = \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j}.$$

In particular, for $j = 2q - 1$,

$$a^{2q-1} = \sum_{k=0}^{2q-1} (-1)^{k-1} \binom{2q-1}{k} \rho^{2k-2q+1} = \rho^{2q-1} - \sum_{k=0}^{2q-2} (-1)^k \binom{2q-1}{k} \rho^{2k-2q+1}.$$

Since $\rho^{4q-2} - \sum_{k=0}^{2q-2} (-1)^k (\rho^{2k} + \rho^{2k+1}) = 0$, we have

$$\rho^{2q-1} = \sum_{k=0}^{2q-2} (-1)^k \rho^{2k-2q+1} + \sum_{k=1}^{2q-1} (-1)^{k-1} \rho^{2k-2q}.$$

Therefore,

$$a^{2q-1} = \sum_{k=1}^{2q-2} (-1)^{k-1} \left[\binom{2q-1}{k} - 1 \right] \rho^{2k-2q+1} + \sum_{k=1}^{2q-1} (-1)^{k-1} \rho^{2k-2q}.$$

The equation we are looking for becomes

$$\sum_{j=0}^{2q-2} \alpha_j \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j} + \sum_{k=1}^{2q-2} (-1)^k \left[\binom{2q-1}{k} - 1 \right] \rho^{2k-2q+1} + \sum_{k=1}^{2q-1} (-1)^k \rho^{2k-2q} = 0. \quad \text{(A.12)}$$

This is the analogue of (A.5). The double sum in the left-hand side is given by (A.4) upon changing q into $q - 1$:

$$\sum_{j=0}^{2q-2} \alpha_j \sum_{k=0}^{j} (-1)^{j+k} \binom{j}{k} \rho^{2k-j} = \sum_{\substack{j \in \mathcal{E}: \\ 0 \leq j \leq 2q-2}} (-1)^{j/2} \binom{j}{j/2} \alpha_j$$

$$+ \sum_{\substack{i \in \mathcal{E}: \\ 1 \leq i \leq 2q-2}} \left[\sum_{\substack{j \in \mathcal{E}: \\ i \leq j \leq 2q-2}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] (\rho^i + \rho^{-i})$$

$$+ \sum_{\substack{i \in \mathcal{O}: \\ 1 \leq i \leq 2q-3}} \left[\sum_{\substack{j \in \mathcal{O}: \\ i \leq j \leq 2q-3}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j \right] (\rho^i - \rho^{-i}). \quad \text{(A.13)}$$

On the other hand,

$$\sum_{k=1}^{2q-1} (-1)^k \rho^{2k-2q} = (-1)^q + \sum_{\substack{i \in \mathcal{E}: \\ 1 \leq i \leq 2q-2}} (-1)^{i/2+q} (\rho^i + \rho^{-i}), \quad \text{(A.14)}$$

and

$$\sum_{k=1}^{2q-2} (-1)^k \left[\binom{2q-1}{k} - 1 \right] \rho^{2k-2q+1} = \sum_{\substack{i \in \mathcal{O}: \\ 1 \leq i \leq 2q-3}} (-1)^{(i-1)/2+q} \left[\binom{2q-1}{(i-1)/2+q} - 1 \right] (\rho^i - \rho^{-i}). \quad \text{(A.15)}$$

Plugging (A.13), (A.14) and (A.15) into (A.12) yields

$$\left[\sum_{\substack{j \in \mathcal{E}: \\ 0 \leq j \leq 2q-2}} (-1)^{j/2} \binom{j}{j/2} \alpha_j + (-1)^q \right] + \sum_{\substack{i \in \mathcal{E}: \\ 1 \leq i \leq 2q-2}} \left[\sum_{\substack{j \in \mathcal{E}: \\ i \leq j \leq 2q-2}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j + (-1)^{i/2+q} \right] (\rho^i + \rho^{-i})$$

$$+ \sum_{\substack{i \in \mathcal{O}: \\ 1 \leq i \leq 2q-3}} \left[\sum_{\substack{j \in \mathcal{O}: \\ i \leq j \leq 2q-3}} (-1)^{(j-i)/2} \binom{j}{(i+j)/2} \alpha_j + (-1)^{(i-1)/2+q} \left[\binom{2q-1}{(i-1)/2+q} - 1 \right] \right] (\rho^i - \rho^{-i}) = 0.$$

This equation is of the form

$$\sum_{i=0}^{2q-2} \beta_i (\rho^i + (-1)^i \rho^{-i}) = 0$$

which is satisfied as soon as the β_i's vanish, that is when the α_j's satisfy

$$
\begin{cases}
\displaystyle\sum_{\substack{j\in\mathcal{E}:\\ i\leq j\leq 2q-2}} (-1)^{j/2}\binom{j}{(i+j)/2}\alpha_j = (-1)^{q-1} & \text{for } i\in\{0,2,4,\ldots,2q-2\},\\[2em]
\displaystyle\sum_{\substack{j\in\mathcal{O}:\\ i\leq j\leq 2q-3}} (-1)^{(j+1)/2}\binom{j}{(i+j)/2}\alpha_j = (-1)^{q}\left[\binom{2q-1}{(i-1)/2+q}-1\right] & \text{for } i\in\{1,3,5,\ldots,2q-3\}.
\end{cases}
$$

With an obvious change of indices, we rewrite this system as

$$
\begin{cases}
\displaystyle\sum_{j=i}^{q-1}(-1)^j\binom{2j}{j-i}\alpha_{2j} = (-1)^{q-1} & \text{for } i\in\{0,1,2,\ldots,q-1\},\\[2em]
\displaystyle\sum_{j=i}^{q-1}(-1)^j\binom{2j-1}{j-i}\alpha_{2j-1} = (-1)^{q}\left[\binom{2q-1}{q-i}-1\right] & \text{for } i\in\{1,2,3,\ldots,q-1\}.
\end{cases}
\tag{A.16}
$$

By introducing the matrices

$$
T_1 = \left[(-1)^j\binom{2j}{j-i}\right]_{0\leq i,j\leq q-1}, \quad T_2 = \left[(-1)^j\binom{2j-1}{j-i}\right]_{1\leq i,j\leq q-1},
$$

$$
A_1 = [\alpha_{2i}]_{0\leq i\leq q-1}, \quad A_2 = [\alpha_{2i-1}]_{1\leq i\leq q-1},
$$

$$
B_1 = (-1)^{q-1}[1]_{0\leq i\leq q-1}, \quad B_2 = (-1)^q\left[\binom{2q-1}{q-i}\right]_{1\leq i\leq q-1} - (-1)^q[1]_{1\leq i\leq q-1},
$$

the system (A.16) can be simply written as $T_1 A_1 = B_1$ and $T_2 A_2 = B_2$. Observe that the matrix T_1 is exactly the same as the T_1 in the previous case, while T_2 is nearly the same as T_2 in the previous case upon changing q into $q-1$.

Proposition A.3 *We have*

$$
A_1 = \left[\binom{q+i-1}{2i}\right]_{0\leq i\leq q-1}, \quad A_2 = -\left[\binom{q+i-1}{2i-1}\right]_{1\leq i\leq q-1}.
$$

PROOF. Referring to Proposition A.1 for T_1^{-1}, we first compute the product $A_1 = T_1^{-1}B_1$:

$$
\left[(-1)^j\frac{2j}{i+j}\binom{i+j}{2i}\right]_{0\leq i,j\leq q-1} \times [(-1)^{q-1}]_{0\leq i\leq q-1}.
$$

The entries of this matrix are

$$
\alpha_{2i} = \sum_{k=i}^{q-1}(-1)^{k+q-1}\frac{2k}{k+i}\binom{k+i}{2i} = \sum_{k=i}^{q-1}(-1)^{k+q-1}\left(2-\frac{2i}{k+i}\right)\binom{k+i}{2i}
$$

$$
= \sum_{k=i}^{q-1}(-1)^{k+q-1}\left[2\binom{k+i}{2i}-\binom{k+i-1}{2i-1}\right].
\tag{A.17}
$$

By using Pascal's rule, we replace $\binom{k+i}{2i}-\binom{k+i-1}{2i-1}$ by $\binom{k+i-1}{2i}$ which yields

$$
\alpha_{2i} = (-1)^{q-1}\sum_{k=i}^{q-1}\left[(-1)^k\binom{k+i}{2i}-(-1)^{k-1}\binom{k+i-1}{2i}\right].
$$

Observing that the above sum is telescopic, we finally obtain the value of α_{2i}:

$$
\alpha_{2i} = \binom{q+i-1}{2i}.
\tag{A.18}
$$

Referring to Proposition A.1 for T_2^{-1} (upon changing q into $q-1$), we now compute the product $A_2 = T_2^{-1}B_2$ which can be split into the difference of two matrices $A_2 = A_2' - A_2''$ because of the structure of B_2, where

$$
A_2' = (-1)^q\left[(-1)^j\frac{2j-1}{i+j-1}\binom{i+j-1}{2i-1}\right]_{1\leq i,j\leq q-1} \times \left[\binom{2q-1}{q-i}\right]_{1\leq i\leq q-1},
$$

$$
A_2'' = (-1)^q\left[(-1)^j\frac{2j-1}{i+j-1}\binom{i+j-1}{2i-1}\right]_{1\leq i,j\leq q-1} \times [1]_{1\leq i\leq q-1}.
$$

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The entries of A_2' are

$$\alpha_{2i-1}' = \sum_{k=i}^{q-1}(-1)^{k+q}\frac{2k-1}{k+i-1}\binom{k+i-1}{2i-1}\binom{2q-1}{q-k}.$$

By expanding the product $T_2^{-1}T_2$ (as in the proof of Proposition A.2) which coincides with the unit matrix of type $(q-1)\times(q-1)$ and changing q into $q+1$ therein (since it works for any q), we obtain the analogue of (A.11), namely for any i such that $1 \le i \le q-1$,

$$\sum_{k=i}^{q}(-1)^{k}\frac{2k-1}{k+i-1}\binom{k+i-1}{2i-1}\binom{2q-1}{q-k} = 0.$$

By isolating the last term in the above sum, we get

$$\sum_{k=i}^{q-1}(-1)^{k}\frac{2k-1}{k+i-1}\binom{k+i-1}{2i-1}\binom{2q-1}{q-k} = (-1)^{q-1}\frac{2q-1}{q+i-1}\binom{q+i-1}{2i-1}$$

and

$$\alpha_{2i-1}' = -\frac{2q-1}{q+i-1}\binom{q+i-1}{2i-1}.$$

On the other hand, the entries of A_2'' are

$$\alpha_{2i-1}'' = \sum_{k=i}^{q-1}(-1)^{k+q}\frac{2k-1}{k+i-1}\binom{k+i-1}{2i-1}.$$

This expression is quite similar to the first sum of (A.17) and can be evaluated in the same manner by writing $\frac{2k-1}{k+i-1} = 2 - \frac{2i-1}{k+i-1}$ and using Pascal's rule. We find the analogue of (A.18):

$$\alpha_{2i-1}'' = -\binom{q+i-2}{2i-1}.$$

Finally, the entries of $A_2 = A_2' - A_2''$ write as

$$\alpha_{2i-1} = \alpha_{2i-1}' - \alpha_{2i-1}'' = -\frac{2q-1}{q+i-1}\binom{q+i-1}{2i-1} + \binom{q+i-2}{2i-1}$$

which can be simplified into

$$\alpha_{2i-1} = -\binom{q+i-1}{2i-1}.$$

The proof of Proposition A.3 is finished. $\qquad\qquad\square$

Consequently, we can state the result below.

Theorem A.2 *Let q be a positive integer and ρ be a root of the equation $r^{4q} = r^{4q-1} + r + 1$ distinct of $\pm i$. The parameter $a = \rho - 1/\rho$ is an algebraic number; it solves the equation*

$$a^{2q-1} = \sum_{i=0}^{q-1}\binom{q+i-1}{2i}a^{2i} - \sum_{i=1}^{q-1}\binom{q+i-1}{2i-1}a^{2i-1}.$$

Examples 5 *For $q \in \{1,2,3,4\}$ ($p \in \{3,7,11,15\}$), the corresponding equation satisfied by a reads*

$$a - 1 = 0,$$
$$a^3 - a^2 + 2a - 1 = 0,$$
$$a^5 - a^4 + 4a^3 - 3a^2 + 3a - 1 = 0,$$
$$a^7 - a^6 + 6a^5 - 5a^4 + 10a^3 - 6a^2 + 4a - 1 = 0.$$

Acknowledgements

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